Math 502 - Complex Analysis

(Analysis B)

Blue Book description: Complex numbers. Holomorphic functions. Cauchy's theorem. Meromorphic functions. Laurent expansions, residue calculus. Conformal maps, topology of the plane.

Course Objectives

The course is devoted to the analysis of differentiable functions of a complex variable. This is a central topic in pure mathematics, as well as a vital computational tool.

Syllabus

1. **Basics.** Geometric description of complex numbers. The complex plane and the Riemann sphere. Conformal mappings. Linear transformations as conformal maps. Representation by complex numbers.

2. **Holomorphic functions.** Definition. Exponential and trigonometric functions. Conformality and the Cauchy-Riemann equations. Relation to harmonic functions. Power series: uniform convergence, Weierstrass' M-test, continuity, integrability, differentiability. Integration along curves, primitives.

3. **Cauchy's Theorem and Applications.** Goursat's theorem. Cauchy's theorem on a disk. Evaluation of integrals. Morera's theorem. Cauchy integral formulas. Cauchy estimates and Liouville's theorem. Fundamental theorem of algebra. Isolated zeros and analytic continuation. Sequences of holomorphic functions. Schwarz reflection principle.

4. **Meromorphic functions.** Zeros and poles. Laurent series. The residue formula for some domains. Jordan and "small arc" lemmas. Computation of integrals by residue calculus. Riemann's theorem on removable singularities. Essential singularities and Casorati-Weierstrass theorem. The argument principle and applications (Rouché's theorem, open mapping theorem, maximum modulus theorem). Gamma and zeta functions.

5. **Plane topology.** Simply and multiply connected domains. Jordan curve theorem (without proof). Homotopies and the winding number. General form of Cauchy's theorem. Roots and logarithm (including branches and cuts). Additional examples of computation of integrals using residues.

6. **Conformal maps.** Elementary conformal maps. Schwarz lemma and automorphisms of the disk and upper-half plane. Fractional linear transformations (cross ratio, behavior of lines and

circles). Normal familes. Montel's theorem and the Riemann mapping theorem.

7. **Other optional topics, if time is available.** Paley-Wiener theorem and functions of exponential type. The prime number theorem. Entire functions and Picards theorem. Riemann surfaces. Elliptic functions.

References

[S] Elias Stein, Rami Shakarchi, *Complex Analysis*, Princeton University Press 2003.